Prepared by:

Date:

Quantum Theory of Scattering.

I. Rutherford Scattering

$$\frac{d\sigma}{d\Omega} = \frac{Z^2 z^2}{16 K^2} \, \frac{\alpha^2 (\hbar c)^2}{\sin^4 \theta/2}$$

K = kinetic energy of the beam.

Total Cross Section $\sigma = \int \frac{d\sigma}{d\Omega} \, d\Omega$

II. 11.1.2 Griffiths

$$\psi(r,\theta) \cong A\left\{ e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r} \right\}$$

for large r.

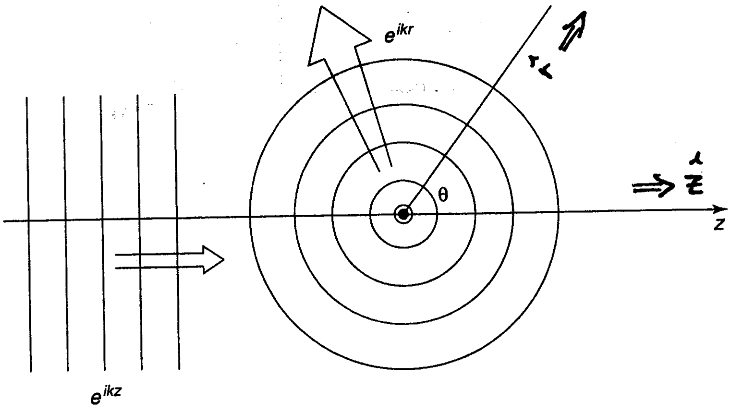


FIGURE 11.4: Scattering of waves; incoming plane wave generates outgoing spherical wave.

$$k = \frac{\sqrt{2mE}}{\hbar}$$

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2$$

Born Approximation

Ae^{ikz} incident plane wave

Integral form of the S.E.

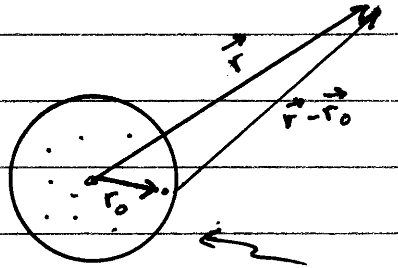
11.4.1 Griffiths

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \, V(\vec{r}_0)\,\psi(\vec{r}_0)\, d^3\vec{r}_0$$

where $\psi_0(\vec{r})$ satisfies the free-particle S.E. $(\nabla^2 + k^2)\psi_0 = 0$

$$|\vec{r}-\vec{r}_0|^2 = r^2 + r_0^2 - 2\vec{r}\cdot\vec{r}_0 \cong r^2\left(1 - \frac{2\vec{r}\cdot\vec{r}_0}{r^2}\right)$$

$$|\vec{r}-\vec{r}_0| \cong r - \hat{r}\cdot\vec{r}_0$$



$$\boxed{\vec{k} = k\hat{r}} \qquad \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \cong \frac{e^{ikr}}{r}\, e^{-i\vec{k}\cdot\vec{r}_0}$$

So, $\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \cong \frac{e^{ikr}}{r}\, e^{-i\vec{k}\cdot\vec{r}_0}$

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Quantum Theory of Scattering

$$\psi(\vec r) \cong A e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec k\cdot\vec r_0}\, V(\vec r_0)\,\psi(\vec r_0)\, d^3\vec r_0$$

So, $f(\theta,\varphi) = \frac{m}{2\pi\hbar^2 A}\int e^{-i\vec k\cdot\vec r_0}\, V(\vec r_0)\,\psi(\vec r_0)\, d^3\vec r_0$

Now, invoke the Born Approximation $\psi(\vec r_0) \underset{\nearrow}{\approx} \psi_0(\vec r_0) = A e^{ikz_0} = A e^{i\vec k'\cdot\vec r_0}$

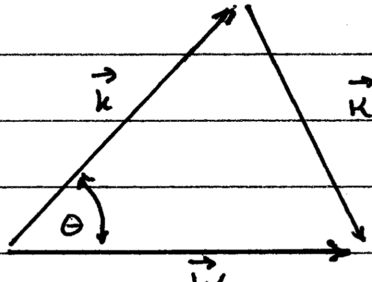
incoming plane wave is not substantially altered

$\vec k' \equiv k\hat z$

$$\boxed{f(\theta,\varphi) \cong -\frac{m}{2\pi\hbar^2}\int e^{i(\vec k'-\vec k)\cdot\vec r_0}\, V(\vec r_0)\, d^3\vec r_0}$$

Eq. 11.79 Griffiths.

$K = 2k\sin\theta/2 = q$



"Elastic Scattering"

$\hbar\vec k' - \hbar\vec k = \hbar\vec q$ momentum transfer

$\vec q \equiv (\vec k' - \vec k) = \vec K$

The interaction V(r) converts the initial wave into a final "scattered" wave

The matrix element $V'_{fi} = \int \psi_f\, V'\, \psi_i\, d^3\vec r_0$

Fermi's Golden Rule ⇒ Decay Rates $R = \frac{2\pi}{\hbar}\left|V'_{fi}\right|^2 \rho(E_f)$

$\rho(E_f)$ = density of final states = $\frac{\text{\# of states}}{\text{unit energy interval}}$ at E_f

Scattering is proportional to the square of the quantity

$$F(\underset{\uparrow\,\text{initial}}{\vec k'}, \underset{\uparrow\,\text{final}}{\vec k}) = \int \psi_f^*\, V(\vec r_0)\,\psi_i\, d^3\vec r_0$$

Quantum Theory of Scattering

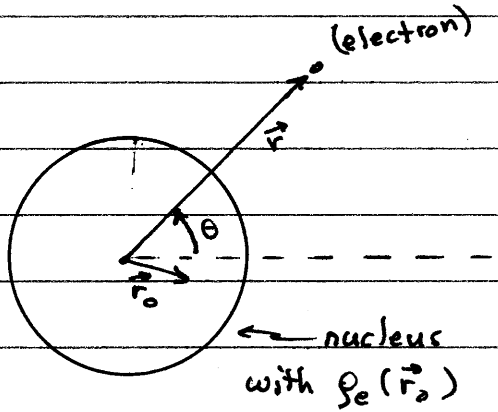
① $$F(\vec{q}) = \int e^{i\vec{q}\cdot\vec{r}} V(\vec{r})\, d^3\vec{r}$$

1.) Normalization is chosen such that $F(0) = 1$

2.) In the nucleus, $V(r)$ depends on the nuclear charge density $Ze\rho_e(\vec{r}_0)$ and $\rho_e(\vec{r}_0)$ is the nuclear charge distribution.
Let the origin of $\vec{r}_0$ be the center of the nucleus.

3.) The P.E. due to dQ at $\vec{r}$ (the location of the beam particle, e.g. electron) is $$dV = -\frac{e\, dQ}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_0|}$$

$$dV = -\frac{Ze^2 \rho_e(\vec{r}_0)\, d^3\vec{r}_0}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_0|}$$



$$V(\vec{r}) = -\frac{Ze^2}{4\pi\varepsilon_0}\int \frac{\rho_e(\vec{r}_0)\, d^3\vec{r}_0}{|\vec{r}-\vec{r}_0|}$$

$\vec{q}\cdot\vec{r}_0 = q r_0 \sin\theta$ and integrating over $\vec{r}_0$ in Eq. 1, we have

$$F(\vec{q}) = \int e^{i\vec{q}\cdot\vec{r}_0} \rho_e(\vec{r}_0)\, d^3\vec{r}_0$$

If $\rho(\vec{r}_0)$ depends only on r_0 and <u>not</u> θ or φ, then

Elastic Scattering

Form Factor
$$\boxed{F(q) = \frac{4\pi}{q}\int \sin q r_0\, \rho_e(r_0)\, r_0\, dr_0}$$

$$q \equiv \frac{2p}{\hbar}\sin\frac{\theta}{2}$$

$F(q)$ is the form factor, and an inverse Fourier transform of $F(q)$ gives us $\rho_e(r_0)$, the charge distribution of the nucleus.

Quantum Scattering

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Recall that scattering light off a circular disk of diameter D is:

$$\theta = \sin^{-1}\left(\frac{1.22\lambda}{D}\right) \qquad p = \frac{h}{\lambda}$$

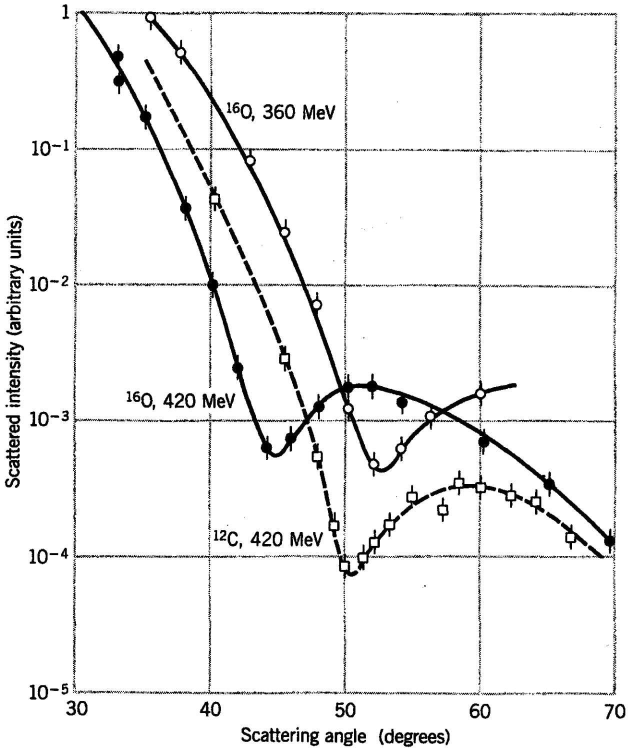

Figure 3.1 Electron scattering from ^{16}O and ^{12}C. The shape of the cross section is somewhat similar to that of diffraction patterns obtained with light waves. The data come from early experiments at the Stanford Linear Accelerator Center (H. F. Ehrenberg et al., *Phys. Rev.* **113**, 666 (1959)).

$\lambda = \frac{hc}{pc}$ E = kinetic energy $\cong$ pc for ultrarelativistic electrons

$$\lambda = \frac{1240 \text{ MeV}\cdot\text{fm}}{E(\text{MeV})} \qquad D = \frac{1.22\lambda}{\sin\theta}$$

↖ 1st minimum

Note: the minimum does not go to zero. The nucleus has a fuzzy edge.

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Quantum Theory of Scattering.

Griffith's 11.79 $$f(\theta,\varphi) \cong -\frac{m}{2\pi\hbar^2}\int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}_0}\,V(r_0)\,d^3r_0$$

For a spherically symmetric potential $V(\vec{r}) = V(r)$

The Born Approximation reduces to: the following:

1.) $q = \vec{k}' - \vec{k}$

2.) let the polar axis for the r_0 integration lie along $\vec{q}$

$$(\vec{k}' - \vec{k})\cdot\vec{r}_0 = q\,r_0\cos\theta_0$$

Then: $$f(\theta) \cong \frac{m}{2\pi\hbar^2}\int e^{iqr_0\cos\theta_0}\,V(r_0)\,r_0^2\sin\theta_0\,dr_0\,d\theta_0\,d\varphi_0$$

$$\int_0^{2\pi} d\varphi_0 = 2\pi \qquad \int_0^{\pi} e^{iqr_0\cos\theta_0}\sin\theta_0\,d\theta_0 = \frac{2\sin qr_0}{qr_0}$$

$$\boxed{f(\theta) \cong \frac{2m}{\hbar^2}\int_0^{\infty} r\,V(r)\sin qr\,dr}$$ (spherical symmetry)

Question: what do we measure in the laboratory? $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$

Question: What is $V(r)$ in our integral if we're scattering high energy electrons off a nucleus? $$V(r) = \frac{-Ze^2}{4\pi\varepsilon_0}\int \frac{\rho_e(r_0)\,d^3r_0}{|\vec{r}-\vec{r}_0|}$$

$$f(\theta) \cong \frac{2m}{\hbar^2}\int_0^{\infty} r\,\underbrace{\left[\frac{-Ze^2}{4\pi\varepsilon_0}\int\frac{\rho_e(r_0)\,d^3r_0}{|\vec{r}-\vec{r}_0|}\right]}_{V(r)}\sin qr\,dr$$

⇒ Charge distribution of the nucleus is $Z\rho_e(r_0)e$ → usually plot $\rho_e(r_0)$

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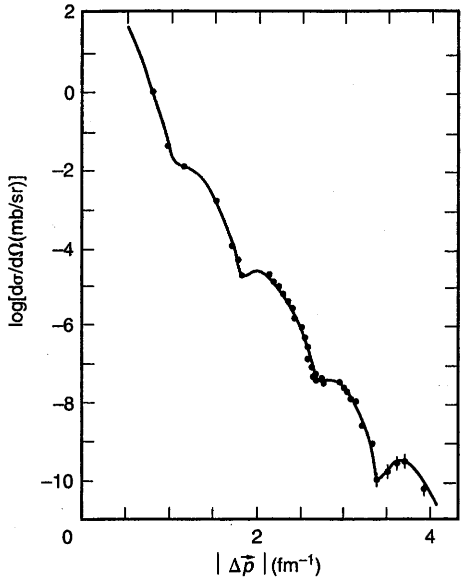
Charge Distribution of the Nucleus.

Look at the $\frac{d\sigma}{d\Omega}$ measurements

Figure 3.6 | Measured Differential Scattering Cross Section for 450 MeV Electrons Incident on ^{58}Ni

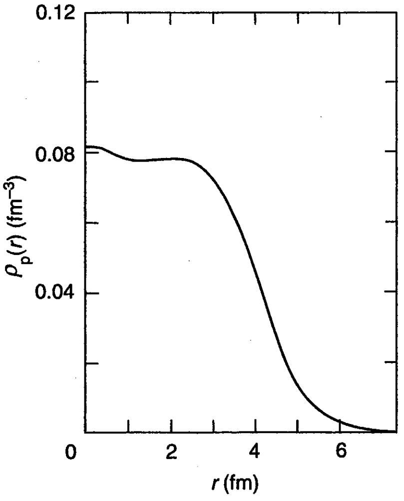
The solid line is a fit using the charge distribution shown in Figure 3.7.



Data from I. Sick *et al., Phys. Rev. Lett. 35* (1975), 910. Copyright 1975 by the American Physical Society.

Inverse Fourier Transform

Figure 3.7 | Charge Distribution for ^{58}Ni Extracted Numerically from the Measurements Shown in Figure 3.6



Data from I. Sick *et al., Phys. Rev. Lett. 35* (1975), 910. Copyright 1975 by the American Physical Society.

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Charge Distribution of the Nucleus

Distribution of Protons:

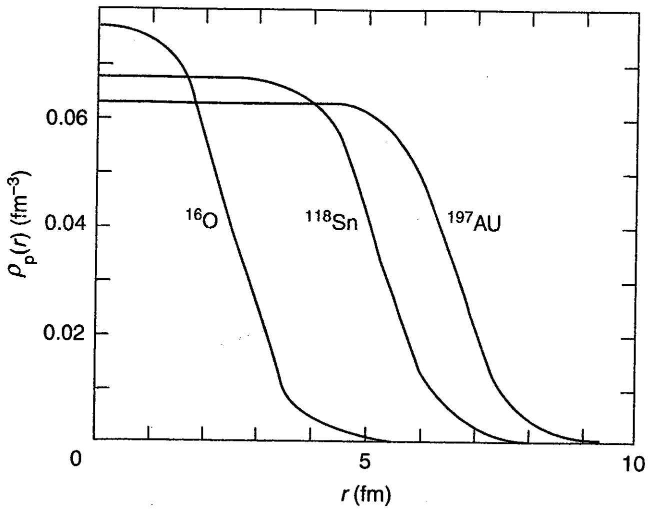
$$\rho_p(r) = \frac{\rho_p^{(0)}}{1 + e^{(r-R)/a}}$$

$$\int \rho_p(\vec{r})\, d^3 r = 4\pi \int_0^\infty \rho_p(r)\, r^2\, dr = Z$$

a = nuclear surface thickness

R = radius of the nucleus

Figure 3.8 | Woods-Saxon Charge Distributions for Some Nuclei

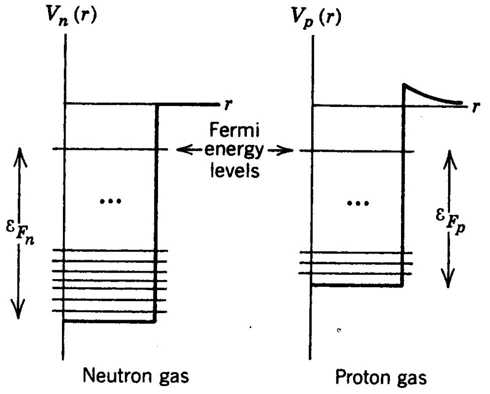


An analysis of $\rho_p(r)$ for several nuclei is shown in Figure 3.8. It is seen that the shape of this function is a reasonable approximation of the results of the model independent analysis shown above. The figure shows some important aspects of the nuclear charge distributions.

1. Larger nuclei have a larger mean diameter.
2. The edge region has a similar width in all nuclei.
3. The charge density at the center is greater in light nuclei than in heavy nuclei.

Figure 14-15

Neutron and proton potential energy wells in the Fermi gas model. The proton potential energy is elevated by the effects of Coulomb repulsion.



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Mass Distribution in the Nucleus

$$\frac{\text{neutron density}}{\text{proton density}} = \frac{\rho_n(r)}{\rho_p(r)} = \frac{N}{Z}$$

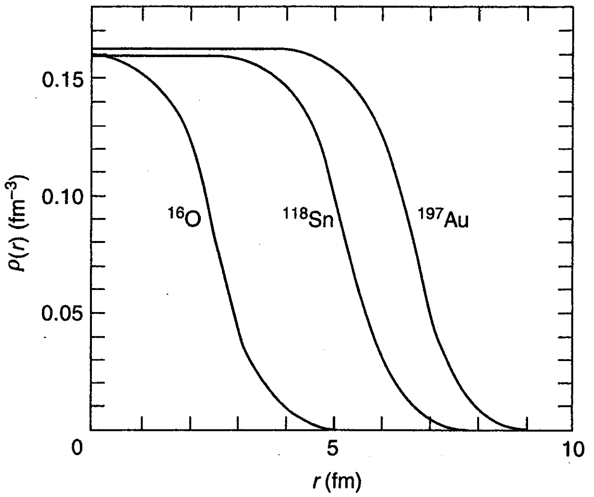
$$\rho(r) = \rho_p(r) + \rho_n(r)$$

Again $\rho_p(r) = \dfrac{\rho_p^{(0)}}{1+e^{(r-R)/a}}$

$$\Rightarrow \rho_{\text{nucleons}}(r) = \rho_p(r)\left[1 + \frac{N}{Z}\right]$$

Figure 3.9 | Woods-Saxon Mass Distributions for Some Nuclei

Note the differences between these distributions and the charge distributions for the same nuclei shown in Figure 3.8.



It is also known that the total density is the sum of the neutron and proton densities

$$\rho(r) = \rho_p(r) + \rho_n(r). \tag{3.21}$$

These two equations can be easily solved to give

$$\rho(r) = \rho_p(r)\left[1 + \frac{N}{Z}\right]. \tag{3.22}$$

Homework: 3.4, 3.6, 3.7, 3.8